

# COBRAIDED SMASH PRODUCT HOM-HOPF ALGEBRAS

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**ABSTRACT.** Let  $(A, \alpha)$  and  $(B, \beta)$  be two Hom-Hopf algebras. In this paper, we construct a class of new Hom-Hopf algebras:  $R$ -smash product  $(A \sharp_R B, \alpha \otimes \beta)$ . Moreover, necessary and sufficient conditions for  $(A \sharp_R B, \alpha \otimes \beta)$  to be a cbraided Hom-Hopf algebra are given.

## 1. INTRODUCTION

Hom-structures (Lie algebras, algebras, coalgebras, Hopf algebras) have been intensively investigated in the literature recently, see [1, 3, 5, 6, 11, 12, 13]. Hom-algebras are generalizations of algebras obtained by a twisting map, which have been introduced for the first time in [5]. The associativity is replaced by Hom-associativity, Hom-coassociativity for a Hom-coalgebra can be considered in a similar way. Also definitions and properties of Hom-bialgebras and Hom-Hopf algebras have been proposed, see [1, 3, 6, 12, 13].

In [1], Caenepeel and Goyvaerts studied the Hom-structures from the point of view of monoidal categories and introduced that Hom-algebras coincide with algebras in a symmetric monoidal category. In [12], Yau presented the notion of cbraided Hom-bialgebras and got the following conclusion: each cbraided Hom-bialgebra comes with solutions of the operator quantum Hom-Yang-Baxter equations, which are twisted analogues of the operator form of the quantum Yang-Baxter equation. Solutions of the Hom-Yang-Baxter equation can be obtained from comodules of suitable cbraided Hom-bialgebras. And in [11], Yau introduced and characterized the concept of module Hom-algebras as a twisted version of usual module algebras.

Let  $H$  be a Hopf algebra and  $A$  an  $H$ -module algebra, then, as we all know, we can construct a new Hopf algebra: smash product  $A \# H$  (see [7] or [8]). The extended forms of smash product can be found in the following literature [2, 4].

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Let  $(H, \beta)$  be a Hom-Hopf algebra and  $(A, \alpha)$  an  $(H, \beta)$ -module Hom-algebra (introduced by Yau in [11]), then it is natural to ask: How to construct smash product Hom-Hopf algebra and when the smash product Hom-Hopf algebra is cobarided?

The purpose of this article is to give a positive answer to the above questions.

This article is organized as follows. In Section 2, we recall some definitions and results which will be used later. In Section 3, instead of constructing smash product Hom-Hopf algebra  $(A \sharp H, \alpha \otimes \beta)$  (see Theorem 3.3.), we give a more general case, so-called  $R$ -smash product Hom-Hopf algebra  $(A \sharp_R B, \alpha \otimes \beta)$  (see Theorem 3.1). We remark that the smash product Hom-Hopf algebra  $(A \sharp H, \alpha \otimes \beta)$  in Theorem 3.3 is different from the one defined by Chen-Wang-Zhang in [3], since here the construction of  $(A \sharp H, \alpha \otimes \beta)$  is based on the concept of the module Hom-algebra introduced by Yau in [11] which differs from the Chen-Wang-Zhang's in [3]. Necessary and sufficient conditions for  $(A \sharp_R B, \alpha \otimes \beta)$  to be a cobarided Hom-Hopf algebra are derived in Section 4 (see Theorem 4.8, Theorem 4.9). And in last section, we give a concrete example.

## 2. PRELIMINARIES

Throughout this paper, we follow the definitions and terminologies in [1, 11, 12], with all algebraic systems supposed to be over the field  $K$ . Given a  $K$ -space  $M$ , we write  $id_M$  for the identity map on  $M$ .

We now recall some useful definitions.

**Definition 2.1** A Hom-algebra is a quadruple  $(A, \mu, 1_A, \alpha)$  (abbr.  $(A, \alpha)$ ), where  $A$  is a  $K$ -linear space,  $\mu : A \otimes A \rightarrow A$  is a  $K$ -linear map,  $1_A \in A$  and  $\alpha$  is an automorphism of  $A$ , such that

$$\begin{aligned} (HA1) \quad & \alpha(aa') = \alpha(a)\alpha(a'); \quad \alpha(1_A) = 1_A, \\ (HA2) \quad & \alpha(a)(a'a'') = (aa')\alpha(a''); \quad a1_A = 1_Aa = \alpha(a) \end{aligned}$$

are satisfied for  $a, a', a'' \in A$ . Here we use the notation  $\mu(a \otimes a') = aa'$ .

**Definition 2.2** A Hom-coalgebra is a quadruple  $(C, \Delta, \varepsilon_C, \beta)$  (abbr.  $(C, \beta)$ ), where  $C$  is a  $K$ -linear space,  $\Delta : C \rightarrow C \otimes C$ ,  $\varepsilon_C : C \rightarrow K$  are  $K$ -linear maps, and  $\beta$  is an automorphism of  $C$ , such that

$$\begin{aligned} (HC1) \quad & \beta(c)_1 \otimes \beta(c)_2 = \beta(c_1) \otimes \beta(c_2); \quad \varepsilon_C \circ \beta = \varepsilon_C \\ (HC2) \quad & \beta(c_1) \otimes c_{21} \otimes c_{22} = c_{11} \otimes c_{12} \otimes \beta(c_2); \quad \varepsilon_C(c_1)c_2 = c_1\varepsilon_C(c_2) = \beta(c) \end{aligned}$$

are satisfied for  $c \in A$ . Here we use the notation  $\Delta(c) = c_1 \otimes c_2$  (summation implicitly understood).

**Remarks** (1) The first equation in  $(HC2)$  is equivalent to

$$c_1 \otimes c_{21} \otimes c_{22} = \beta^{-1}(c_{11}) \otimes c_{12} \otimes \beta(c_2) \quad (1)$$

and

$$c_{11} \otimes c_{12} \otimes c_2 = \beta(c_1) \otimes c_{21} \otimes \beta^{-1}(c_{22}), \quad (2)$$

respectively.

(2) By (1), (2) and  $(HC2)$ , we have

$$c_{11} \otimes c_{12} \otimes c_{21} \otimes c_{22} = \beta(c_1) \otimes \beta^{-1}(c_{211}) \otimes \beta^{-1}(c_{212}) \otimes c_{22}. \quad (3)$$

**Definition 2.3** A Hom-bialgebra is a sextuple  $(H, \mu, 1_H, \Delta, \varepsilon, \gamma)$  (abbr.  $(H, \gamma)$ ), where  $(H, \mu, 1_H, \gamma)$  is a Hom-algebra and  $(H, \Delta, \varepsilon, \gamma)$  is a Hom-coalgebra, such that  $\Delta$  and  $\varepsilon$  are morphisms of Hom-algebras, i.e.

$$\begin{aligned} \Delta(hh') &= \Delta(h)\Delta(h'); \quad \Delta(1_H) = 1_H \otimes 1_H, \\ \varepsilon(hh') &= \varepsilon(h)\varepsilon(h'); \quad \varepsilon(1_H) = 1. \end{aligned}$$

Furthermore, if there exists a linear map  $S : H \longrightarrow H$  such that

$$S(h_1)h_2 = h_1S(h_2) = \varepsilon(h)1_H \text{ and } S(\gamma(h)) = \gamma(S(h)),$$

then we call  $(H, \mu, 1_H, \Delta, \varepsilon, \gamma, S)$  (abbr.  $(H, \gamma, S)$ ) a Hom-Hopf algebra.

Let  $(H, \gamma)$  and  $(H', \gamma')$  be two Hom-bialgebras. The linear map  $f : H \longrightarrow H'$  is called a Hom-bialgebra map if  $f \circ \gamma = \gamma' \circ f$  and at the same time  $f$  is a bialgebra map in the usual sense.

**Definition 2.4** Let  $(A, \beta)$  be a Hom-algebra. A left  $(A, \beta)$ -Hom-module is a triple  $(M, \triangleright, \alpha)$ , where  $M$  is a linear space,  $\triangleright : A \otimes M \longrightarrow M$  is a linear map, and  $\alpha$  is an automorphism of  $M$ , such that

$$\begin{aligned} (HM1) \quad & \alpha(a \triangleright m) = \beta(a) \triangleright \alpha(m), \\ (HM2) \quad & \beta(a) \triangleright (a' \triangleright m) = (aa') \triangleright \alpha(m); \quad 1_A \triangleright m = \alpha(m) \end{aligned}$$

are satisfied for  $a, a' \in A$  and  $m \in M$ .

**Remarks** (1) It is obvious that  $(A, \mu, \beta)$  is a left  $(A, \beta)$ -Hom-module.

(2) When  $\beta = id_A$  and  $\alpha = id_M$ , a left  $(A, \beta)$ -Hom-module is the usual left  $A$ -module.

**Definition 2.5** Let  $(H, \beta)$  be a Hom-bialgebra and  $(A, \alpha)$  a Hom-algebra. If  $(A, \triangleright, \alpha)$  is a left  $(H, \beta)$ -Hom-module and for all  $h \in H$  and  $a, a' \in A$ ,

$$(HMA1) \quad \beta^2(h) \triangleright (aa') = (h_1 \triangleright a)(h_2 \triangleright a'),$$

$$(HMA2) \quad h \triangleright 1_A = \varepsilon_H(h)1_A,$$

then  $(A, \triangleright, \alpha)$  is called an  $(H, \beta)$ -module Hom-algebra.

**Remarks** (1) When  $\alpha = id_A$  and  $\beta = id_H$ , an  $(H, \beta)$ -module Hom-algebra is the usual  $H$ -module algebra.

(2) Similar to the case of Hopf algebras, in [13], Yau concluded that the Eq.(HMA1) is satisfied if and only if  $\mu_A$  is a morphism of  $H$ -modules for suitable  $H$ -module structures on  $A \otimes A$  and  $A$ , respectively.

(3) If  $\beta^2 = id$  in (HMA1), then we can get (6.1) in [3]. So two definitions of module Hom-algebra are different which leads to the differentness of smash product Hom-Hopf algebra in Theorem 3.3 and in Definition 6.2 in [3].

**Definition 2.6** A cogenerated Hom-Hopf algebra is a octuple  $(H, \mu, 1_H, \Delta, \varepsilon, S, \alpha, \sigma)$  (abbr.  $(H, \alpha, \sigma)$ ) in which  $(H, \mu, 1_H, \Delta, \varepsilon, S, \alpha)$  is a Hom-Hopf algebra and  $\sigma$  is a bilinear form on  $H$  (i.e.,  $\sigma \in Hom(H \otimes H, K)$ ), satisfying the following axioms (for all  $h, g, l \in H$ ):

$$\begin{aligned} (CHA1) \quad & \sigma(h, 1_H) = \sigma(1_H, h) = \varepsilon(h), \\ (CHA2) \quad & \sigma(hg, \alpha(l)) = \sigma(\alpha(h), l_1)\sigma(\alpha(g), l_2), \\ (CHA3) \quad & \sigma(\alpha(h), gl) = \sigma(h_1, \alpha(l))\sigma(h_2, \alpha(g)), \\ (CHA4) \quad & \sigma(h_1, g_1)h_2g_2 = g_1h_1\sigma(h_2, g_2), \\ (CHA5) \quad & \sigma(\alpha(h), \alpha(g)) = \sigma(h, g). \end{aligned}$$

In this case,  $\sigma$  is called the Hom-cobraiding form.

**Remarks** (1) When  $\alpha = id_H$ , a cogenerated Hom-Hopf algebra is exactly the usual cogenerated (or coquasitriangular) Hopf algebra.

(2) It is slightly different from the definition in [12] or [13]. Here we replace the Hom-bialgebra by Hom-Hopf algebra and also add another two conditions (CHA1) and (CHA5). Similar to the Hopf algebra setting, the Hom-cobraiding form  $\sigma$  in Definition 2.6 is invertible.

(3) Based on Yau's results in [12], each cogenerated Hom-Hopf algebra comes with solutions of the operator quantum Hom-Yang-Baxter equations, which are twisted analogues of the operator form of the quantum Yang-Baxter equation.

Next, we generalize the concept of skew pairing to Hom-setting.

**Definition 2.7** Let  $(A, \alpha, S_A)$  and  $(B, \beta, S_B)$  be two Hom-Hopf algebras,  $\vartheta \in Hom(A \otimes B, K)$  a bilinear form. A Hom-skew pairing is a triple  $(A, B, \vartheta)$

such that

$$\begin{aligned}
(SP1) \quad & \vartheta(a, 1_B) = \varepsilon_A(a); \quad \vartheta(1_A, b) = \varepsilon_B(b), \\
(SP2) \quad & \vartheta(aa', \beta(b)) = \vartheta(\alpha(a), b_1) \vartheta(\alpha(a'), b_2), \\
(SP3) \quad & \vartheta(\alpha(a), bb') = \vartheta(a_1, \beta(b')) \vartheta(a_2, \beta(b)), \\
(SP4) \quad & \vartheta(\alpha(a), \beta(b)) = \vartheta(a, b),
\end{aligned}$$

where  $a, a' \in A$  and  $b, b' \in B$ .

**Remarks** (1) When  $\alpha = id_A$  and  $\beta = id_B$ , we can get the usual skew pairing.

(2) If  $(H, \alpha, \sigma)$  is a cobraided Hom-Hopf algebra, then  $(H, H, \sigma)$  is a Hom-skew pairing.

(3)  $\vartheta$  is (convolution) invertible with  $\vartheta^{-1}(a, b) = \vartheta(S_A(a), b)$ .

### 3. SMASH PRODUCT HOM-HOPF ALGEBRA

In this section, we introduce a class of new Hom-Hopf algebras:  $R$ -smash product  $A \sharp_R B$ , generalizing  $R$ -smash product studied in [2]. As a special case, Hom-smash product is derived based on the structure of module Hom-algebra introduced by Yau in [11] or [13].

Let  $A$  and  $B$  be two linear spaces,  $R : B \otimes A \longrightarrow A \otimes B$  a linear map. In the following, we write  $R(b \otimes a) = \sum a_R \otimes b_R$  for all  $a \in A$  and  $b \in B$ , and the notations  $\sum a_r \otimes b_r$ ,  $\sum a_{R'} \otimes b_{R'}$  are the copies of  $\sum a_R \otimes b_R$ . As usual, we omit the summation sign “ $\sum$ ”.

**Theorem 3.1** Let  $(A, \mu_A, 1_A, \alpha)$  and  $(B, \mu_B, 1_B, \beta)$  be two Hom-algebras,  $R : B \otimes A \longrightarrow A \otimes B$  a linear map such that for all  $a \in A, b \in B$ ,

$$\alpha(a)_R \otimes \beta(b)_R = \alpha(a_R) \otimes \beta(b_R). \quad (4)$$

Then  $(A \sharp_R B, \alpha \otimes \beta)$  ( $A \sharp_R B = A \otimes B$  as a linear space) with the multiplication

$$(a \otimes b)(a' \otimes b') = a\alpha^{-1}(a')_R \otimes \beta^{-1}(b_R)b',$$

where  $a, a' \in A, b, b' \in B$ , and unit  $1_A \otimes 1_B$  becomes a Hom-algebra if and only if the following conditions hold:

$$\begin{aligned}
(C1) \quad & a_R \otimes 1_{BR} = \alpha(a) \otimes 1_B; \quad 1_{AR} \otimes b_R = 1_A \otimes \beta(b), \\
(C2) \quad & \alpha(a)_R \otimes (bb')_R = a_{Rr} \otimes \beta^{-1}(\beta(b)_r)b'_R, \\
(C3) \quad & \alpha((aa')_R) \otimes \beta(b)_R = \alpha(a_R)\alpha(a')_r \otimes b_{Rr},
\end{aligned}$$

where  $a, a' \in A, b, b' \in B$ .

We call this Hom-algebra  $R$ -smash product Hom-algebra and denote it by  $(A \sharp_R B, \alpha \otimes \beta)$ .

**Proof** ( $\Leftarrow$ ) For all  $a, a', a'' \in A$ ,  $b, b', b'' \in B$ , firstly, we prove that (HA1) holds. In fact, one can get

$$\begin{aligned}
(\alpha \otimes \beta)((a \otimes b)(a' \otimes b')) &= \frac{\alpha(a\alpha^{-1}(a')_R) \otimes \beta(\beta^{-1}(b_R)b')}{\stackrel{(HA1)}{=}} \alpha(a)\alpha(\alpha^{-1}(a')_R) \otimes b_R\beta(b') \\
&\stackrel{(4)}{=} \alpha(a)a'_R \otimes \beta^{-1}(\beta(b)_R)\beta(b') \\
&= ((\alpha \otimes \beta)(a \otimes b))((\alpha \otimes \beta)(a' \otimes b'))
\end{aligned}$$

and

$$(\alpha \otimes \beta)(1_A \otimes 1_B) = \alpha(1_A) \otimes \beta(1_B) \stackrel{(HA1)}{=} 1_A \otimes 1_B.$$

Secondly, we compute the condition (HA2) as follows.

$$\begin{aligned}
&(\alpha(a) \otimes \beta(b))((a' \otimes b')(a'' \otimes b'')) \\
&= \alpha(a)\alpha^{-1}(a'\alpha^{-1}(a'')_R)_r \otimes \beta^{-1}(\beta(b)_r)(\beta^{-1}(b'_R)b'') \\
&\stackrel{(C3)}{=} \alpha(a)\alpha^{-1}(\alpha(\alpha^{-1}(a')_r)\alpha^{-1}(a'')_{RR'}) \otimes \beta^{-1}(b_{rR'})(\beta^{-1}(b'_R)b'') \\
&= \frac{\alpha(a)(\alpha^{-1}(a')_r\alpha^{-1}(\alpha^{-1}(a'')_{RR'}))}{\stackrel{(HA2)}{=}} \otimes \beta^{-1}(b_{rR'})(\beta^{-1}(b'_R)b'') \\
&\stackrel{(HA2)}{=} (a\alpha^{-1}(a')_r)\alpha^{-1}(a'')_{RR'} \otimes \beta^{-1}(\beta^{-1}(b_{rR'})b'_R)\beta(b'') \\
&\stackrel{(C2)}{=} (a\alpha^{-1}(a')_r)a''_R \otimes \beta^{-1}((\beta^{-1}(b_r)b')_R)\beta(b'') \\
&= ((a \otimes b)(a' \otimes b'))(\alpha(a'') \otimes \beta(b''))
\end{aligned}$$

and

$$\begin{aligned}
(a \otimes b)(1_A \otimes 1_B) &= a\alpha^{-1}(1_A)_R \otimes \beta^{-1}(b_R)1_B \\
&\stackrel{(HA1)}{=} a1_{AR} \otimes \beta^{-1}(b_R)1_B \\
&\stackrel{(C1)}{=} a1_A \otimes b1_B \\
&\stackrel{(HA2)}{=} \alpha(a) \otimes \beta(b).
\end{aligned}$$

Similarly,  $(1_A \otimes 1_B)(a \otimes b) = \alpha(a) \otimes \beta(b)$  holds.

( $\Rightarrow$ ) By (HA2), we have

$$1_A\alpha^{-1}(a)_R \otimes \beta^{-1}(1_{BR})b = \alpha(a) \otimes \beta(b), \quad (5)$$

$$a\alpha^{-1}(1_A) \otimes \beta^{-1}(b_R)1_B = \alpha(a) \otimes \beta(b) \quad (6)$$

and

$$\begin{aligned}
&\alpha(a)\alpha^{-1}(a'\alpha^{-1}(a'')_R)_r \otimes \beta^{-1}(\beta(b)_r)(\beta^{-1}(b'_R)b'') \\
&= (a\alpha^{-1}(a')_r)a''_R \otimes \beta^{-1}((\beta^{-1}(b_r)b')_R)\beta(b''). \quad (7)
\end{aligned}$$

Let  $b = 1_B$  and  $a = 1$  in Eqs.(5) and (6), respectively, we can get (C1).

Let  $a = a' = 1_A$  and  $b'' = 1_B$  in Eq.(7) and by (C1), then (C2) holds.

Likewise, (C3) can be obtained by letting  $a = 1_A$  and  $b' = b'' = 1_B$  in Eq.(7).  $\square$

When  $\alpha = id_A$  and  $\beta = id_B$ , we have

**Example 3.2**([2]) Let  $(A, \mu_A, 1_A)$  and  $(B, \mu_B, 1_B)$  be two algebras,  $R : B \otimes A \longrightarrow A \otimes B$  a linear map. Then  $A \#_R B$  ( $A \#_R B = A \otimes B$  as a linear space) with the multiplication

$$(a \otimes b)(a' \otimes b') = aa'_R \otimes b_R b',$$

where  $a, a' \in A, b, b' \in B$ , and unit  $1_A \otimes 1_B$  becomes an algebra if and only if the following conditions hold:

- (1)  $a_R \otimes 1_{BR} = a \otimes 1_B; 1_{AR} \otimes b_R = 1_A \otimes b,$
- (2)  $a_R \otimes (bb')_R = a_{Rr} \otimes b_r b'_R,$
- (3)  $(aa')_R \otimes b_R = a_R a'_r \otimes b_{Rr},$

where  $a, a' \in A, b, b' \in B$ .

**Theorem 3.3** Let  $(H, \beta)$  be a Hom-bialgebra and  $(A, \triangleright, \alpha)$  an  $(H, \beta)$ -module Hom-algebra. Then  $(A \sharp H, \alpha \otimes \beta)$  ( $A \sharp H = A \otimes H$  as a linear space) with the multiplication

$$(a \otimes h)(a' \otimes h') = a(h_1 \triangleright \alpha^{-1}(a')) \otimes \beta^{-1}(h_2)h',$$

where  $a, a' \in A, h, h' \in H$ , and unit  $1_A \otimes 1_H$  is a Hom-algebra, we call it smash product Hom-algebra denoted by  $(A \sharp H, \alpha \otimes \beta)$ .

**Proof** Define  $R : H \otimes A \longrightarrow A \otimes H$  by

$$R(h \otimes a) = h_1 \triangleright a \otimes h_2, \forall a \in A, h \in H.$$

Firstly, for all  $a \in A$  and  $h \in H$ ,

$$\begin{aligned} \alpha(a)_R \otimes \beta(h)_R &= \underline{\beta(h)_1 \triangleright \alpha(a)} \otimes \underline{\beta(h)_2} \\ &\stackrel{(HC1)}{=} \underline{\beta(h_1) \triangleright \alpha(a)} \otimes \beta(h_2) \\ &\stackrel{(HM1)}{=} \alpha(h_1 \triangleright a) \otimes \beta(h_2) = \alpha(a_R) \otimes \beta(h_R), \end{aligned}$$

so Eq.(4) holds.

Secondly, we have

$$a_R \otimes 1_{HR} = 1 \triangleright a \otimes 1_H \stackrel{(HM2)}{=} \alpha(a) \otimes 1_H$$

and

$$1_{AR} \otimes h_R = h_1 \triangleright 1_A \otimes h_2 \stackrel{(HMA2)}{=} 1_A \otimes \varepsilon(h_1)h_2 \stackrel{(HC2)}{=} 1_A \otimes \beta(h).$$

Thirdly, we verify that the conditions (C2) and (C3) are satisfied. For all  $a, a' \in A, h, h' \in B$ ,

$$\begin{aligned}
\alpha(a)_R \otimes (hh')_R &= (hh')_1 \triangleright \alpha(a) \otimes (hh')_2 \\
&= \underline{(h_1 h'_1) \triangleright \alpha(a)} \otimes h_2 h'_2 \\
&\stackrel{(HM2)}{=} \underline{\beta(h_1) \triangleright (h'_1 \triangleright a)} \otimes \underline{h_2 h'_2} \\
&\stackrel{(HC1)}{=} \beta(h)_1 \triangleright (h'_1 \triangleright a) \otimes \beta^{-1}(\beta(h)_2) h'_2 \\
&= a_{Rr} \otimes \beta^{-1}(\beta(h)_r) h'_R
\end{aligned}$$

and

$$\begin{aligned}
\alpha((aa')_R) \otimes \beta(h)_R &= \alpha(\underline{\beta(h)_1 \triangleright (aa')}) \otimes \underline{\beta(h)_2} \\
&\stackrel{(HC1)}{=} \underline{\alpha(\beta(h_1) \triangleright (aa'))} \otimes \beta(h_2) \\
&\stackrel{(HM1)}{=} \beta^2(h_1) \triangleright \underline{\alpha(aa')} \otimes \beta(h_2) \\
&\stackrel{(HA1)}{=} \underline{\beta^2(h_1) \triangleright (\alpha(a)\alpha(a'))} \otimes \beta(h_2) \\
&\stackrel{(HMA1)}{=} \underline{(h_{11} \triangleright \alpha(a))(h_{12} \triangleright \alpha(a'))} \otimes \underline{\beta(h_2)} \\
&\stackrel{(HC2)}{=} \underline{(\beta(h_1) \triangleright \alpha(a))(h_{21} \triangleright \alpha(a'))} \otimes h_{22} \\
&\stackrel{(HM1)}{=} \alpha(h_1 \triangleright a)(h_{21} \triangleright \alpha(a')) \otimes h_{22} \\
&= \alpha(a_R)\alpha(a')_r \otimes h_{Rr}.
\end{aligned}$$

Thus we complete the proof.  $\square$

**Remarks** (1) The smash product Hom-Hopf algebra  $(A \sharp H, \alpha \otimes \beta)$  is different from the one defined by Chen-Wang-Zhang in [3], since here the construction of  $(A \sharp B, \alpha \otimes \beta)$  is based on the concept of the module Hom-algebra introduced by Yau in [11], while two of conditions (6.1), (6.2) in the module Hom-algebra in [3] are same to the case of Hopf algebra.

(2) When  $\alpha = id_A$  and  $\beta = id_H$ , we can get the usual smash product algebra  $A \# H$  (see [7, 8]).

**Lemma 3.4** Let  $(C, \alpha)$  and  $(D, \beta)$  be two Hom-coalgebras. Then  $(C \otimes D, \alpha \otimes \beta)$  is a Hom-coalgebra with the following comultiplication and counit

$$\begin{aligned}
\Delta(c \otimes d) &= c_1 \otimes d_1 \otimes c_2 \otimes d_2, \\
\varepsilon(c \otimes d) &= \varepsilon_C(c) \varepsilon_D(d),
\end{aligned}$$

in which  $c \in C$  and  $d \in D$ . We call it tensor product Hom-coalgebra.

**Proof** Straightforward.  $\square$

**Theorem 3.5** Let  $(A, \alpha, S_A)$  and  $(B, \beta, S_B)$  be two Hom-Hopf algebras,  $R : B \otimes A \rightarrow A \otimes B$  a linear map. Then the  $R$ -smash product Hom-algebra



$(A \sharp_R B, \alpha \otimes \beta)$  equipped with the tensor product Hom-coalgebra structure becomes a Hom-bialgebra if and only if  $R$  is a coalgebra map, i.e.

$$\begin{aligned} a_{R1} \otimes b_{R1} \otimes a_{R2} \otimes b_{R2} &= a_{1R} \otimes b_{1R} \otimes a_{2r} \otimes b_{2r}, \\ \varepsilon_A(a_R) \varepsilon_B(b_R) &= \varepsilon_A(a) \varepsilon_B(b), \end{aligned}$$

where  $a \in A$ ,  $b \in B$ .

Furthermore,  $R$ -smash product Hom-bialgebra  $(A \sharp_R B, \alpha \otimes \beta)$  is a Hom-Hopf algebra with antipode  $\bar{S}$  defined by

$$\bar{S}(a \otimes b) = \alpha^{-1}(S_A(a))_R \otimes \beta^{-1}(S_B(b))_R.$$

**Proof** We only prove that  $\bar{S}$  is an antipode of  $(A \sharp_R B, \alpha \otimes \beta)$ . The rest is straightforward by direct computation. For all  $a \in A$  and  $b \in B$ ,

$$\begin{aligned} (\bar{S} * id_{A \sharp_R B})(a \otimes b) &= (\alpha^{-1}(S_A(a_1))_R \otimes \beta^{-1}(S_B(b_1))_R)(a_2 \otimes b_2) \\ &= \alpha^{-1}(S_A(a_1))_R \alpha^{-1}(a_2)_r \otimes \beta^{-1}(\beta^{-1}(S_B(b_1))_R)_r b_2 \\ &\stackrel{(4)}{=} \alpha^{-1}(S_A(a_1))_R \alpha^{-1}(a_{2r}) \otimes \beta^{-2}(S_B(b_1)_{Rr}) b_2 \\ &\stackrel{(HA1)}{=} \alpha^{-1}(\alpha(\alpha^{-1}(S_A(a_1))_R) a_{2r}) \otimes \beta^{-2}(S_B(b_1)_{Rr}) b_2 \\ &\stackrel{(C3)}{=} \alpha^{-1}(S_A(a_1) a_2)_R \otimes \beta^{-2}(\beta(S_B(b_1))_R) b_2 \\ &= \underline{1_{AR} \varepsilon_A(a)} \otimes \beta^{-2}(\beta(S_B(b_1))_R) b_2 \\ &\stackrel{(C1)}{=} 1_A \varepsilon_A(a) \otimes S_B(b_1) b_2 \\ &= 1_A \otimes 1_B \varepsilon_A(a) \varepsilon_B(b) \\ &= 1_A \otimes 1_B \bar{\varepsilon}(a \otimes b) \end{aligned}$$

and

$$\begin{aligned} (id_{A \sharp_R B} * \bar{S})(a \otimes b) &= (a_1 \otimes b_1)(\alpha^{-1}(S_A(a_2))_R \otimes \beta^{-1}(S_B(b_2))_R) \\ &= a_1 \alpha^{-1}(\alpha^{-1}(S_A(a_2))_R)_r \otimes \beta^{-1}(b_{1r}) \beta^{-1}(S_B(b_2))_R \\ &\stackrel{(4)}{=} a_1 \alpha^{-2}(S_A(a_2))_{Rr} \otimes \beta^{-1}(b_{1r}) \beta^{-1}(S_B(b_2))_R \\ &= a_1 \alpha^{-1}(\alpha^{-1}(S_A(a_2)))_{Rr} \\ &\quad \otimes \beta^{-1}(\beta(\beta^{-1}(b_1))_r) \beta^{-1}(S_B(b_2))_R \\ &\stackrel{(C2)}{=} a_1 \alpha^{-1}(S_A(a_2))_R \otimes \beta^{-1}(b_1 S_B(b_2)) \\ &= a_1 \alpha^{-1}(S_A(a_2))_R \otimes \underline{1_{BR} \varepsilon_B(b)} \\ &\stackrel{(C1)}{=} a_1 S_A(a_2) \otimes 1_B \varepsilon_B(b) \\ &= 1_A \otimes 1_B \varepsilon_A(a) \varepsilon_B(b) \\ &= 1_A \otimes 1_B \bar{\varepsilon}(a \otimes b), \end{aligned}$$

while

$$\begin{aligned}
\bar{S}(\alpha(a) \otimes \beta(b)) &= \alpha^{-1}(S_A(\alpha(a)))_R \otimes \beta^{-1}(S_B(\beta(b)))_R \\
&= \alpha^{-1}(\alpha(S_A(a)))_R \otimes \beta^{-1}(\beta(S_B(b)))_R \\
&= S_A(a)_R \otimes \beta^{-1}(\beta(S_B(b)))_R \\
&\stackrel{(4)}{=} \alpha(\alpha^{-1}(S_A(a)))_R \otimes S_B(b)_R \\
&= (\alpha \otimes \beta)(\bar{S}(a \otimes b)),
\end{aligned}$$

finishing the proof.  $\square$

When  $\alpha = id_A$  and  $\beta = id_B$ , we have

**Example 3.6**([2]) Let  $A$  and  $B$  be two Hopf algebras. Then the twisted tensor product algebra  $A \#_R B$  equipped with the usual tensor product coalgebra structure is a bialgebra if and only if  $R$  is a coalgebra map.

Furthermore, twisted tensor product bialgebra  $A \#_R B$  is a Hopf algebra with antipode  $S_{A \#_R B}$  defined by

$$S_{A \#_R B}(a \otimes b) = S_A(a)_R \otimes S_B(b)_R.$$

**Theorem 3.7** Let  $(H, \beta)$  be a Hom-Hopf algebra and  $(A, \triangleright, \alpha)$  an  $(H, \beta)$ -module Hom-algebra. Then the smash product Hom-algebra  $(A \sharp H, \alpha \otimes \beta)$  endowed with the tensor product Hom-coalgebra structure becomes a Hom-bialgebra if and only if

$$(h \triangleright a)_1 \otimes (h \triangleright a)_2 = (h_1 \triangleright a_1) \otimes (h_2 \triangleright a_2); \quad \varepsilon_A(h \triangleright a) = \varepsilon_A(a)\varepsilon_H(h) \quad (8)$$

and

$$h_1 \otimes h_2 \triangleright a = h_2 \otimes h_1 \triangleright a. \quad (9)$$

Moreover, smash product Hom-bialgebra  $(A \sharp H, \alpha \otimes \beta)$  is a Hom-Hopf algebra with antipode

$$S_{A \sharp H}(a \otimes h) = S_H(h)_1 \triangleright \alpha^{-1}(S_A(a)) \otimes \beta^{-1}(S_H(h)_2).$$

**Proof** Let  $R(h \otimes a) = h_1 \triangleright a \otimes h_2, \forall a \in A, h \in H$  in Theorem 3.5. Then  $R$  is a coalgebra map if and only if

$$(h_1 \triangleright a)_1 \otimes h_{21} \otimes (h_1 \triangleright a)_2 \otimes h_{22} = h_{11} \triangleright a_1 \otimes h_{12} \otimes h_{21} \triangleright a_2 \otimes h_{22} \quad (10)$$

and

$$\varepsilon_A(h \triangleright a) = \varepsilon_A(a)\varepsilon_H(h).$$

And by Eq.(3) and (HC1), it is easy to obtain that the first equation in Eq.(8) and Eq.(9) are equivalent to Eq.(10).  $\square$

**Remarks** (1) Let  $(H, \beta)$  be a Hom-Hopf algebra. Assume that  $(A, \triangleright, \alpha)$  is a Hom-coalgebra and an  $(H, \beta)$ -Hom-module satisfying Eq.(8). Then we call  $(A, \triangleright, \alpha)$  an  $(H, \beta)$ -module Hom-coalgebra.

When  $\alpha = id_A$  and  $\beta = id_H$ , then an  $(H, \beta)$ -module Hom-coalgebra is exactly the module coalgebra in the usual case (see [7]).

(2) Theorem 3.7 is the Hom-version of the usual smash product Hopf algebra (see [7]).

#### 4. COBRAIDED HOM-HOPF ALGEBRA

In this section, necessary and sufficient conditions for smash product Hom-Hopf algebra to be cbraided are given.

**Proposition 4.1** Let  $(A \sharp_R B, \alpha \otimes \beta)$  be a  $R$ -smash product Hom-Hopf algebra. Define

$$i : A \longrightarrow A \sharp_R B, \quad i(a) = a \otimes 1_B; \quad j : B \longrightarrow A \sharp_R B, \quad j(b) = 1_A \otimes b$$

for all  $a \in A$  and  $b \in B$ . Then  $i$  and  $j$  are both Hom-bialgebra maps.

**Proof** Straightforward.  $\square$

Let  $(A \sharp_R B, \alpha \otimes \beta)$  be a  $R$ -smash product Hom-Hopf algebra, and  $\sigma : A \sharp_R B \otimes A \sharp_R B \longrightarrow K$  a bilinear form. Define

$$\begin{aligned} \tau : A \otimes A &\longrightarrow K, \quad \tau(a, a') = \sigma(i \otimes i)(a \otimes a'), \\ v : B \otimes B &\longrightarrow K, \quad v(b, b') = \sigma(j \otimes j)(b \otimes b'), \\ \varphi : A \otimes B &\longrightarrow K, \quad \varphi(a, b) = \sigma(i \otimes j)(a \otimes b), \\ \psi : B \otimes A &\longrightarrow K, \quad \psi(b, a) = \sigma(j \otimes i)(b \otimes a), \end{aligned}$$

where  $a, a' \in A$  and  $b, b' \in B$ .

The following two lemmas are obvious.

**Lemma 4.2** Let  $(A \sharp_R B, \alpha \otimes \beta)$  be a  $R$ -smash product Hom-Hopf algebra. If  $\sigma$  satisfies (CHA1), then for  $a \in A$  and  $b \in B$ ,

- (1)  $\tau(1_A, a) = \tau(a, 1_A) = \varepsilon_A(a)$ ,
- (2)  $v(b, 1_B) = v(1_B, b) = \varepsilon_B(b)$ ,
- (3)  $\varphi(1_A, b) = \varepsilon_B(b)$ ;  $\varphi(a, 1_B) = \varepsilon_A(a)$ ,
- (4)  $\psi(1_B, a) = \varepsilon_A(a)$ ;  $\psi(b, 1_A) = \varepsilon_B(b)$ .

**Lemma 4.3** Let  $(A \sharp_R B, \alpha \otimes \beta)$  be a  $R$ -smash product Hom-Hopf algebra. If  $\sigma$  satisfies (CHA5) for  $\alpha \otimes \beta$ , then, for  $a, a' \in A$  and  $b, b' \in B$ ,

- (1)  $\tau(\alpha(a), \alpha(a')) = \tau(a, a')$ ,
- (2)  $v(\beta(b), \beta(b')) = v(b, b')$ ,
- (3)  $\varphi(\alpha(a), \beta(b)) = \varphi(a, b)$ ,
- (4)  $\psi(\beta(b), \alpha(a)) = \psi(b, a)$ .

**Lemma 4.4** Let  $(A \sharp_R B, \alpha \otimes \beta, \sigma)$  be a cobraided  $R$ -smash product Hom-Hopf algebra. Then, for all  $a, a' \in A$  and  $b, b' \in B$ ,

$$\sigma(\alpha(a) \otimes \beta(b), \alpha(a') \otimes \beta(b')) = \varphi(a_1, b'_1) \tau(a_2, a'_1) v(b_1, b'_2) \psi(b_2, a'_2). \quad (11)$$

**Proof** By  $(CHA2)$  and  $(CHA3)$ , for all  $a, a', a'', a''' \in A, b, b', b'', b''' \in B$ , we have

$$\begin{aligned} & \sigma(a\alpha^{-1}(a')_R \otimes \beta^{-1}(b_R)b', a''\alpha^{-1}(a''')_r \otimes \beta^{-1}(b''_r)b''') \\ &= \sigma(a_1 \otimes b_1, a'''_1 \otimes b'''_1) \sigma(a_2 \otimes b_2, a''_1 \otimes b''_1) \\ & \quad \times \sigma(a'_1 \otimes b'_1, a'''_2 \otimes b'''_2) \sigma(a'_2 \otimes b'_2, a''_2 \otimes b''_2). \end{aligned}$$

Let  $a' = a''' = 1_A$  and  $b = b'' = 1_B$  in the above equation, then we can get (11).  $\square$

**Lemma 4.5** Let  $(A \sharp_R B, \alpha \otimes \beta, \sigma)$  be a cobraided  $R$ -smash product Hom-Hopf algebra. Then, for all  $a, a' \in A$  and  $b, b' \in B$ ,

$$\begin{aligned} (D1) \quad & \varphi(\alpha(\alpha^{-1}(a)_R), b_1) v(b'_R, b_2) = v(\beta(b'), b_1) \varphi(\alpha(a), b_2), \\ (D2) \quad & \tau(\alpha(\alpha^{-1}(a)_R), a'_1) \psi(b_R, a'_2) = \psi(\beta(b), a'_1) \tau(\alpha(a), a'_2), \\ (D3) \quad & v(b_1, b'_R) \psi(b_2, \alpha(\alpha^{-1}(a)_R)) = \psi(b_1, \alpha(a) v(b_2, \beta(b')), \\ (D4) \quad & \varphi(a_1, b_R) \tau(a_2, \alpha(\alpha^{-1}(a')_R)) = \tau(a_1, \alpha(a') \varphi(a_2, \beta(b)), \\ (D5) \quad & \psi(b_1, a_1) (\alpha(\alpha^{-1}(a_2)_R) \otimes b_{2R}) = (\alpha(a_1) \otimes \beta(b_1)) \psi(b_2, a_2), \\ (D6) \quad & \varphi(a_1, b_1) (\alpha(a_2) \otimes \beta(b_2)) = (\alpha(\alpha^{-1}(a_1)_R) \otimes b_{1R}) \varphi(a_2, b_2). \end{aligned}$$

**Proof** By  $(CHA2)$ , for all  $a, a', a'' \in A, b, b', b'' \in B$ , we can obtain

$$\begin{aligned} & \sigma(a\alpha^{-1}(a')_R \otimes \beta^{-1}(b_R)b', \alpha(a'') \otimes \beta(b'')) \\ &= \sigma(\alpha(a) \otimes \beta(b), a''_1 \otimes b''_1) \sigma(\alpha(a') \otimes \beta(b'), a''_2 \otimes b''_2). \end{aligned} \quad (12)$$

Let  $a = 1_A$  and  $b' = b'' = 1_B$  in Eq.(12), then  $(C1)$  holds by Eq.(11). Similarly, setting  $a = a'' = 1_A$  and  $b' = 1_B$  in Eq.(12), then we can get  $(C2)$  by Eq.(11).

By  $(CHA3)$ , for all  $a, a', a'' \in A, b, b', b'' \in B$ , we have

$$\begin{aligned} & \sigma(\alpha(a) \otimes \beta(b), a'\alpha^{-1}(a'')_R \otimes \beta^{-1}(b'_R)b'') \\ &= \sigma(a_1 \otimes b_1, \alpha(a'') \otimes \beta(b'')) \sigma(a_2 \otimes b_2, \alpha(a') \otimes \beta(b')). \end{aligned} \quad (13)$$

$(C3)$  can be obtained by letting  $a = a' = 1_A$  and  $b'' = 1_B$  in Eq.(13) and by Eq.(11). Likewise, one gets  $(C4)$  by putting  $a' = 1_A$  and  $b = b'' = 1_B$  in Eq.(13) and by Eq.(11).

By  $(CHA4)$ , for all  $a, a' \in A, b, b' \in B$ , we have

$$\begin{aligned} & \sigma(a_1 \otimes b_1, a'_1 \otimes b'_1) (a_2 \alpha^{-1}(a'_2)_R \otimes \beta^{-1}(b_{2R})b'_2) \\ &= (a'_1 \alpha^{-1}(a_1)_R \otimes \beta^{-1}(b'_{1R})b_1) \sigma(a_2 \otimes b_2, a'_2 \otimes b'_2). \end{aligned} \quad (14)$$

Let  $a = 1_A$  and  $b' = 1_B$  in Eq.(14), then we get (C5). And (C6) is derived by letting  $a' = 1_A$  and  $b = 1_B$  in Eq.(14).  $\square$

**Lemma 4.6** Given the cobrading  $\sigma$  on a  $R$ -smash product Hom-Hopf algebra  $(A \sharp_R B, \alpha \otimes \beta)$ , consider the induced maps  $\tau, v, \varphi$  and  $\psi$ . Then

- (1)  $(A, \alpha, \tau)$  and  $(B, \beta, v)$  are cobrained Hom-Hopf algebras,
- (2)  $(A, B, \varphi)$  and  $(B, A, \psi)$  are Hom-skew pairings.

**Proof** (1) Set  $b = b' = b'' = 1_B$  in Eq.(12) and Eq.(13), we can get (CHA2) and (CHA3) for  $\tau$ , respectively. (CHA4) can be derived by letting  $b = b' = 1_B$  in Eq.(14), then by Lemma 4.2 and Lemma 4.3,  $(A, \alpha, \tau)$  is a cobrained Hom-Hopf algebra. Similarly, we can prove that  $(B, \beta, v)$  is a cobrained Hom-Hopf algebra.

(2) Let  $a'' = 1_A$  and  $b = b' = 1_B$  in Eq.(12),  $a' = a'' = 1_A$  and  $b = 1_B$  in Eq.(13), (SP2) and (SP3) can be obtained for  $\varphi$ , respectively. Then  $(A, B, \varphi)$  is a Hom-skew pairing by Lemma 4.2 and Lemma 4.3. The rest of (2) can be similarly demonstrated.  $\square$

**Lemma 4.7** Let  $(A \sharp_R B, \alpha \otimes \beta)$  be a  $R$ -smash product Hom-Hopf algebra. If there exist forms  $\tau : A \otimes A \rightarrow K$ ,  $\varphi : A \otimes B \rightarrow K$ ,  $\psi : B \otimes A \rightarrow K$ , and  $v : B \otimes B \rightarrow K$  such that

- (1)  $(A, \alpha, \tau)$  and  $(B, \beta, v)$  are cobrained Hom-Hopf algebras,
- (2)  $(A, B, \varphi)$  and  $(B, A, \psi)$  are Hom-skew pairings,
- (3) The conditions (D1) – (D6) in Lemma 4.5 hold.

Then  $(A \sharp_R B, \alpha \otimes \beta, \sigma)$  is a cobrained Hom-Hopf algebra with the cobrained structure given by

$$\sigma(\alpha(a) \otimes \beta(b), \alpha(a') \otimes \beta(b')) = \varphi(a_1, b'_1) \tau(a_2, a'_1) v(b_1, b'_2) \psi(b_2, a'_2),$$

for  $a, a' \in A$  and  $b, b' \in B$ .

**Proof** It is obvious that  $\sigma$  satisfies (CHA1) and (CHA5).

Next, we show that (CHA2) holds for  $\sigma$ . For all  $a, a', a'' \in A, b, b', b'' \in B$ ,

$$\begin{aligned} & \sigma((a \otimes b)(a' \otimes b'), \alpha(a'') \otimes \beta(b'')) \\ = & \sigma(a\alpha^{-1}(a')_R \otimes \beta^{-1}(b_R)b', \alpha(a'') \otimes \beta(b'')) \\ = & \varphi(\alpha^{-1}(a\alpha^{-1}(a')_R)_1, b''_1) \tau(\alpha^{-1}(a\alpha^{-1}(a')_R)_2, a''_1) \\ & \times v(\beta^{-1}(\beta^{-1}(b_R)b')_1, b''_2) \psi(\beta^{-1}(\beta^{-1}(b_R)b')_2, a''_2) \\ \stackrel{(HA1)(HC1)}{=} & \varphi(\alpha^{-1}(a_1)\alpha^{-1}(\alpha^{-1}(a')_{R1}), b''_1) \tau(\alpha^{-1}(a_2)\alpha^{-1}(\alpha^{-1}(a')_{R2}), a''_1) \\ & \times v(\beta^{-2}(b_{R1})\beta^{-1}(b'_1), b''_2) \psi(\beta^{-2}(b_{R2})\beta^{-1}(b'_2), a''_2) \\ \stackrel{(CHA2)(SP2)}{=} & \varphi(a_1, \beta^{-1}(b''_{11})) \varphi(\alpha^{-1}(a')_{R1}, \beta^{-1}(b''_{12})) \tau(a_2, \alpha^{-1}(a''_{11})) \\ & \times \tau(\alpha^{-1}(a')_{R2}, \alpha^{-1}(a''_{12})) v(\beta^{-1}(b_{R1}), \beta^{-1}(b''_{21})) v(b'_1, \beta^{-1}(b''_{22})) \end{aligned}$$

$$\begin{aligned}
& \times \psi(\beta^{-1}(\underline{b_{R2}}), \alpha^{-1}(\underline{a''_{21}})) \psi(b'_2, \alpha^{-1}(\underline{a''_{22}})) \\
= & \varphi(a_1, \underline{\beta^{-1}(b''_{11})}) \varphi(\alpha^{-1}(a')_{1R}, \underline{\beta^{-1}(b''_{12})}) \tau(a_2, \underline{\alpha^{-1}(a''_{11})}) \\
& \times \tau(\alpha^{-1}(a')_{2r}, \underline{\alpha^{-1}(a''_{12})}) \times v(\beta^{-1}(b_{1R}), \underline{\beta^{-1}(b''_{21})}) v(b'_1, \underline{\beta^{-1}(b''_{22})}) \\
& \times \psi(\beta^{-1}(b_{2r}), \underline{\alpha^{-1}(a''_{21})}) \psi(b'_2, \underline{\alpha^{-1}(a''_{22})}) \\
\stackrel{(3)}{=} & \varphi(a_1, b'_1) \varphi(\underline{\alpha^{-1}(a')_{1R}}, \underline{\beta^{-2}(b''_{211})}) \tau(a_2, a''_1) \\
& \times \tau(\underline{\alpha^{-1}(a')_{2r}}, \underline{\alpha^{-2}(a''_{211})}) v(\underline{\beta^{-1}(b_{1R})}, \underline{\beta^{-2}(b''_{212})}) v(b'_1, \underline{\beta^{-1}(b''_{22})}) \\
& \times \psi(\underline{\beta^{-1}(b_{2r})}, \underline{\alpha^{-2}(a''_{212})}) \psi(b'_2, \underline{\alpha^{-1}(a''_{22})}) \\
\stackrel{(4)(HC1)}{=} & \varphi(a_1, b'_1) \varphi(\alpha(\alpha^{-1}(\alpha^{-1}(a')_1)_R), \beta^{-2}(b''_{21})_1) \tau(a_2, a''_1) \\
& \times \tau(\alpha(\alpha^{-1}(\alpha^{-1}(a')_2)_r), \alpha^{-2}(a''_{21})_1) \underline{v(\beta^{-1}(b_1)_R, \beta^{-2}(b''_{21})_2)} \\
& \times v(b'_1, \underline{\beta^{-1}(b''_{22})}) \underline{\psi(\beta^{-1}(b_2)_r, \alpha^{-2}(a''_{21})_2)} \psi(b'_2, \underline{\alpha^{-1}(a''_{22})}) \\
\stackrel{(D1)(D2)}{=} & \varphi(a_1, \underline{b'_1}) \varphi(a'_1, \underline{\beta^{-2}(b''_{21})_2}) \tau(a_2, \underline{a''_1}) \tau(a'_2, \underline{\alpha^{-2}(a''_{21})_2}) \\
& \times v(b_1, \underline{\beta^{-2}(b''_{21})_1}) v(b'_1, \underline{\beta^{-1}(b''_{22})}) \psi(b_2, \underline{\alpha^{-2}(a''_{21})_1}) \psi(b'_2, \underline{\alpha^{-1}(a''_{22})}) \\
\stackrel{(3)}{=} & \varphi(a_1, \underline{\beta^{-1}(b''_{11})}) \varphi(a'_1, \underline{\beta^{-1}(b''_{21})}) \tau(a_2, \underline{\alpha^{-1}(a''_{11})}) \tau(a'_2, \underline{\alpha^{-1}(a''_{21})}) \\
& \times v(b_1, \underline{\beta^{-1}(b''_{12})}) v(b'_1, \underline{\beta^{-1}(b''_{22})}) \psi(b_2, \underline{\alpha^{-1}(a''_{12})}) \psi(b'_2, \underline{\alpha^{-1}(a''_{22})}) \\
\stackrel{(HC1)}{=} & \varphi(a_1, \beta^{-1}(b''_{11})_1) \tau(a_2, \alpha^{-1}(a''_{11})_1) v(b'_1, \beta^{-1}(b''_{21})_2) \psi(b'_2, \alpha^{-1}(a''_{21})_2) \\
& \times \varphi(a'_1, \beta^{-1}(b''_{21})_1) \tau(a'_2, \alpha^{-1}(a''_{21})_1) v(b_1, \beta^{-1}(b''_{12})_2) \psi(b_2, \alpha^{-1}(a''_{12})_2) \\
= & \sigma(\alpha(a) \otimes \beta(b), a''_1 \otimes b'_1) \sigma(\alpha(a') \otimes \beta(b'), a''_2 \otimes b'_2).
\end{aligned}$$

(CHA3) for  $\sigma$  can be proved by similar method.

And we check (CHA4) as follows. For all  $a, a' \in A$  and  $b, b' \in B$ ,

$$\begin{aligned}
& \sigma(a_1 \otimes b_1, a'_1 \otimes b'_1)(a_2 \otimes b_2)(a'_2 \otimes b'_2) \\
= & u(\alpha^{-1}(a_1)_1, \beta^{-1}(b'_1)_1) \tau(\alpha^{-1}(a_1)_2, \alpha^{-1}(a'_1)_1) v(\beta^{-1}(b_1)_1, \beta^{-1}(b'_1)_2) \\
& \times \psi(\beta^{-1}(b_1)_2, \alpha^{-1}(a'_1)_2) (a_2 \alpha^{-1}(a'_2)_R \otimes \beta^{-1}(b_{2R}) b'_2) \\
\stackrel{(HC1)}{=} & \varphi(\alpha^{-1}(a_{11}), \beta^{-1}(b'_{11})) \tau(\alpha^{-1}(a_{12}), \alpha^{-1}(a'_{11})) v(\beta^{-1}(b_{11}), \beta^{-1}(b'_{12})) \\
& \times \psi(\beta^{-1}(b_{12}), \alpha^{-1}(a'_{12})) (a_2 \alpha^{-1}(a'_2)_R \otimes \beta^{-1}(b_{2R}) b'_2) \\
& \times \psi(\beta^{-1}(b_1)_2, \alpha^{-1}(a'_1)_2) (a_2 \alpha^{-1}(a'_2)_R \otimes \beta^{-1}(b_{2R}) b'_2) \\
\stackrel{(2)}{=} & \varphi(a_1, b'_1) \tau(\alpha^{-1}(a_{21}), a'_1) v(b_1, \beta^{-1}(b'_{21})) \psi(\beta^{-1}(b_{21}), \alpha^{-1}(a'_{21})) \\
& \times (\alpha^{-1}(a_{22}) \alpha^{-2}(a'_{22})_R \otimes \beta^{-1}(\beta^{-1}(b_{22})_R) \beta^{-1}(b'_{22})) \\
\stackrel{(HC1)}{=} & \varphi(a_1, b'_1) \tau(\alpha^{-1}(a_2)_1, a'_1) v(b_1, \beta^{-1}(b'_2)_1) \psi(\beta^{-1}(b_2)_1, \alpha^{-1}(a'_2)_1) \\
& \times (\alpha^{-1}(a_2)_2 \underline{\alpha^{-1}(\alpha^{-1}(a'_2)_2)_R} \otimes \underline{\beta^{-1}(\beta^{-1}(b_2)_{2R})} \beta^{-1}(b'_2)_2)
\end{aligned}$$

$$\begin{aligned}
& \stackrel{(D5)}{=} \varphi(a_1, b'_1) \tau(\alpha^{-1}(a_2)_1, a'_1) v(b_1, \beta^{-1}(b'_2)_1) \psi(\beta^{-1}(b_2)_2, \alpha^{-1}(a'_2)_2) \\
& \quad \times (\alpha^{-1}(a_2)_2 \alpha^{-1}(a'_2)_1 \otimes \beta^{-1}(b_2)_1 \beta^{-1}(b'_2)_2) \\
& \stackrel{(1)(HC1)}{=} \varphi(a_1, b'_1) \tau(\alpha^{-1}(a_2)_1, \alpha^{-1}(a'_1)_1) \underline{v(\beta^{-1}(b_1)_1, \beta^{-1}(b'_2)_1)} \psi(b_2, a'_2) \\
& \quad \times (\alpha^{-1}(a_2)_2 \alpha^{-1}(a'_1)_2 \otimes \beta^{-1}(b_1)_2 \beta^{-1}(b'_2)_2) \\
& \stackrel{(CHA4)}{=} u(a_1, b'_1) \tau(\alpha^{-1}(a_2)_2, \alpha^{-1}(a'_1)_2) v(\beta^{-1}(b_1)_2, \beta^{-1}(b'_2)_2) \psi(b_2, a'_2) \\
& \quad \times (\alpha^{-1}(a'_1)_1 \alpha^{-1}(a_2)_1 \otimes \beta^{-1}(b'_2)_1 \beta^{-1}(b_1)_1) \\
& \stackrel{(1)(HC1)}{=} \varphi(\alpha^{-1}(a_1)_1, \beta^{-1}(b'_1)_1) \tau(a_2, \alpha^{-1}(a'_1)_2) v(\beta^{-1}(b_1)_2, b'_2) \psi(b_2, a'_2) \\
& \quad \times (\alpha^{-1}(a'_1)_1 \alpha^{-1}(a_1)_2 \otimes \beta^{-1}(b'_1)_2 \beta^{-1}(b_1)_1) \\
& \stackrel{(D6)}{=} \varphi(\alpha^{-1}(a_1)_2, \beta^{-1}(b'_1)_2) \tau(a_2, \alpha^{-1}(a'_1)_2) v(\beta^{-1}(b_1)_2, b'_2) \psi(b_2, a'_2) \\
& \quad \times (\alpha^{-1}(a'_1)_1 \alpha^{-1}(\alpha^{-1}(a_1)_1)_R \otimes \beta^{-1}(\beta^{-1}(b'_1)_{1R}) \beta^{-1}(b_1)_1) \\
& \stackrel{(2)(3)}{=} (a'_1 \alpha^{-1}(a_1)_R \otimes \beta^{-1}(b'_{1R}) b_1) \varphi(\alpha^{-1}(a_2)_1, \beta^{-1}(b'_1)_2) \\
& \quad \times \tau(\alpha^{-1}(a_2)_2, \alpha^{-1}(a'_2)_1) v(\beta^{-1}(b_2)_1, \beta^{-1}(b'_2)_2) \psi(\beta^{-1}(b_2)_2, \alpha^{-1}(a'_2)_2) \\
& = (a'_1 \otimes b'_1)(a_1 \otimes b_1) \sigma(a_2 \otimes b_2, a'_2 \otimes b'_2).
\end{aligned}$$

Therefore,  $(A \sharp_R B, \alpha \otimes \beta, \sigma)$  is a cobraided Hom-Hopf algebra.  $\square$

Thus it follows from Lemmas 4.2-4.7 that we have

**Theorem 4.8** *R-smash product Hom-Hopf algebra  $(A \sharp_R B, \alpha \otimes \beta)$  is cobraided if and only if there exist forms  $\tau : A \otimes A \rightarrow K$ ,  $\varphi : A \otimes B \rightarrow K$ ,  $\psi : B \otimes A \rightarrow K$ , and  $v : B \otimes B \rightarrow K$  such that  $(A, \alpha, \tau)$  and  $(B, \beta, v)$  are cobraided Hom-Hopf algebras,  $(A, B, \varphi)$  and  $(B, A, \psi)$  are Hom-skew pairings, the conditions (D1) – (D6) in Lemma 4.5 hold. Moreover, the cobraided structure  $\sigma$  on  $(A \sharp_R B, \alpha \otimes \beta)$  has a decomposition*

$$\sigma(\alpha(a) \otimes \beta(b), \alpha(a') \otimes \beta(b')) = \varphi(a_1, b'_1) \tau(a_2, a'_1) v(b_1, b'_2) \psi(b_2, a'_2).$$

**Theorem 4.9** *Smash product Hom-Hopf algebra  $(A \sharp H, \alpha \otimes \beta)$  is cobraided if and only if there exist forms  $\tau : A \otimes A \rightarrow K$ ,  $\varphi : A \otimes H \rightarrow K$ ,  $\psi : H \otimes A \rightarrow K$ , and  $v : H \otimes H \rightarrow K$  such that  $(A, \alpha, \tau)$  and  $(H, \beta, v)$  are cobraided Hom-Hopf algebras,  $(A, H, \varphi)$  and  $(H, A, \psi)$  are Hom-skew pairings, the conditions (D1)' – (D6)' below hold. For all  $a, a' \in A$  and  $h, h' \in B$ ,*

$$\begin{aligned}
(D1)' \quad & \varphi(\beta(h'_1) \triangleright a, h_1) v(h'_2, h_2) = v(\beta(h'), h_1) \varphi(\alpha(a), h_2), \\
(D2)' \quad & \tau(\beta(h_1) \triangleright a, a'_1) \psi(h_2, a'_2) = \psi(\beta(h), a'_1) \tau(\alpha(a), a'_2), \\
(D3)' \quad & v(h_1, h'_2) \psi(h_2, \beta(h'_1) \triangleright a) = \psi(h_1, \alpha(a) v(h_2, \beta(h'))), \\
(D4)' \quad & \varphi(a_1, h_2) \tau(a_2, \beta(h_1) \triangleright a') = \tau(a_1, \alpha(a') \varphi(a_2, \beta(h))),
\end{aligned}$$

$$(D5)' \quad \psi(h_1, a_1)(\beta(h_{21}) \triangleright a_2 \otimes h_{22}) = (\alpha(a_1) \otimes \beta(h_1))\psi(h_2, a_2),$$

$$(D6)' \quad \varphi(a_1, h_1)(\alpha(a_2) \otimes \beta(h_2)) = (\beta(h_{11}) \triangleright a_1 \otimes h_{12})\varphi(a_2, h_2).$$

Moreover, the cobraided structure  $\sigma'$  on  $(A\sharp H, \alpha \otimes \beta)$  has a decomposition

$$\sigma'(\alpha(a) \otimes \beta(h), \alpha(a') \otimes \beta(h')) = \varphi(a_1, h'_1)\tau(a_2, a'_1)v(h_1, h'_2)\psi(h_2, a'_2).$$

**Proof** Let  $R(h \otimes a) = h_1 \triangleright a \otimes h_2, \forall a \in A, h \in H$  in Theorem 4.8.  $\square$

## 5. APPLICATIONS

In this section, we give the applications of the main results in Sec.3 and 4 to a concrete example.

The following result is clear.

**Lemma 5.1** Let  $K\mathbb{Z}_2 = K\{1, a\}$  be Hopf group algebra (see [9]). Then  $(K\mathbb{Z}_2, id_{K\mathbb{Z}_2}, v)$  is a cobraided Hom-Hopf algebra, where  $v$  is given by

$$\begin{array}{c|cc} v & 1 & a \\ \hline 1 & 1 & 1 \\ a & 1 & -1 \end{array}.$$

Let  $T_{2,-1} = K\{1, g, x, gx | g^2 = 1, x^2 = 0, xg = -gx\}$  be Taft's Hopf algebra (see [10]), its coalgebra structure and antipode are given by

$$\Delta(g) = g \otimes g, \Delta(x) = x \otimes g + 1 \otimes x, \Delta(gx) = gx \otimes 1 + g \otimes gx;$$

$$\varepsilon(g) = 1, \varepsilon(x) = 0, \varepsilon(gx) = 0;$$

and

$$S(g) = g, S(x) = gx, S(gx) = -x.$$

Define a linear map  $\alpha: T_{2,-1} \longrightarrow T_{2,-1}$  by

$$\alpha(1) = 1, \alpha(g) = g, \alpha(x) = kx, \alpha(gx) = kgx$$

where  $0 \neq k \in K$ . Then  $\alpha$  is an automorphism of Hopf algebras.

So we can get a Hom-Hopf algebra  $H_\alpha = (T_{2,-1}, \alpha \circ \mu_{T_{2,-1}}, 1_{T_{2,-1}}, \Delta_{T_{2,-1}} \circ \alpha, \varepsilon_{T_{2,-1}}, \alpha)$  (see [6]).

**Lemma 5.2** Let  $H_\alpha$  be the Hom-Hopf algebra defined as above. Then  $(H_\alpha, \alpha, \tau)$  is a cobraided Hom-Hopf algebra, where  $\tau$  is given by

$$\begin{array}{c|cccc} \tau & 1 & g & x & gx \\ \hline 1 & 1 & 1 & 0 & 0 \\ g & 1 & -1 & 0 & 0 \\ x & 0 & 0 & 0 & 0 \\ gx & 0 & 0 & 0 & 0 \end{array}.$$



$$\begin{aligned} 1_{K\mathbb{Z}_2} \triangleright 1_{H_\alpha} &= 1_{H_\alpha}, \quad 1_{K\mathbb{Z}_2} \triangleright g = g, \\ 1_{K\mathbb{Z}_2} \triangleright x &= kx, \quad 1_{K\mathbb{Z}_2} \triangleright gx = kgx, \\ a \triangleright 1_{H_\alpha} &= 1_{H_\alpha}, \quad a \triangleright g = g, \\ a \triangleright x &= -kx, \quad a \triangleright gx = -kgx, \end{aligned}$$

Furthermore,  $(H_\alpha \bowtie K\mathbb{Z}_2, \alpha \otimes id_{K\mathbb{Z}_2})$  with the tensor product Hom-coalgebra becomes a Hom-Hopf algebra, where the antipode  $\bar{S}$  is given by

$$\begin{aligned}\bar{S}(1_{H_\alpha} \otimes 1_{K\mathbb{Z}_2}) &= 1_{H_\alpha} \otimes 1_{K\mathbb{Z}_2}, \quad \bar{S}(1_{H_\alpha} \otimes a) = 1_{H_\alpha} \otimes a, \\ \bar{S}(g \otimes 1_{K\mathbb{Z}_2}) &= g \otimes 1_{K\mathbb{Z}_2}, \quad \bar{S}(g \otimes a) = g \otimes a, \\ \bar{S}(x \otimes 1_{K\mathbb{Z}_2}) &= -gx \otimes 1_{K\mathbb{Z}_2}, \quad \bar{S}(x \otimes a) = -gx \otimes a, \\ \bar{S}(gx \otimes 1_{K\mathbb{Z}_2}) &= x \otimes 1_{K\mathbb{Z}_2}, \quad \bar{S}(gx \otimes a) = x \otimes a.\end{aligned}$$

$$\begin{array}{c|cc} \varphi & 1 & a \\ \hline 1 & 1 & 1 \\ g & 1 & -1 \\ x & 0 & 0 \\ gx & 0 & 0 \end{array} \quad \begin{array}{c|ccc} \psi & 1 & g & x & gx \\ \hline 1 & 1 & 1 & 0 & 0 \\ a & 1 & -1 & 0 & 0 \end{array}.$$

□

$\sigma$	$1 \otimes 1$	$1 \otimes a$	$g \otimes 1$	$g \otimes a$	$x \otimes 1$	$x \otimes a$	$gx \otimes 1$	$gx \otimes a$
$1 \otimes 1$	1	1	1	1	0	0	0	0
$1 \otimes a$	1	-1	-1	1	0	0	0	0
$g \otimes 1$	1	-1	-1	1	0	0	0	0
$g \otimes a$	1	1	1	1	0	0	0	0
$x \otimes 1$	0	0	0	0	0	0	0	0
$x \otimes a$	0	0	0	0	0	0	0	0
$gx \otimes 1$	0	0	0	0	0	0	0	0
$gx \otimes a$	0	0	0	0	0	0	0	0

**Proof** It is easy to prove that the conditions  $(D1)' - (D6)'$  hold. And by Lemma 5.1, 5.2, 5.4 and Theorem 4.9, we can finish the proof.  $\square$

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